

# Synchronization of Chaotic Fractional-Order Lu-Lu Systems with Active Sliding Mode Control

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## ABSTRACT

*Synchronization of chaotic and Lu system has been done using the active sliding mode control strategy. Regarding the synchronization task as a control problem, fractional order mathematics is used to express the system and active sliding mode for synchronization. It has been shown that, not only the performance of the proposed method is satisfying with an acceptable level of control signal, but also a rather simple stability analysis is performed. The latter is usually a complicated task for nonlinear chaotic systems.*

**KEYWORDS:** fractional calculus, fractional order active sliding mode controller, synchronization, Lu-Lu

## 1. INTRODUCTION

Chaos synchronization is an important topic in the nonlinear science. Generally speaking, the synchronization phenomenon has the following feature: the trajectories of two systems (master and slave systems) are identical notwithstanding starting from different initial conditions. However, slight errors of initial conditions, for chaos dynamical systems, will lead to completely different trajectories. Therefore, how to control two chaos systems to be synchronized has received a great deal of interest in the past decades.

Fractional calculus is a 300-year-old topic, but its applications to physics and engineering are just a recent focus of interest. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [1], electrode-electrolyte

polarization, and electromagnetic waves [2]. Most recently, many researchers begin to investigate the chaotic dynamics of fractional-order systems. In Ref.[3], it has been shown that Chua's system of 2.7 order can produce a chaotic attractor. Ref.[4] pointed out that chaos can exist in the 2.4 order Rossler system and hyper chaos can also exist in the fractional-order Rossler system with order as low as 3.8. Ref.[5] studied the chaotic behaviors of 2.7-order unified chaotic system. Moreover, in the last decade it has been found that chaotic attractors indeed exist in fractional-order systems [6-10].

The question of stability is very important especially in control theory. In the field of fractional order control systems, there are many challenging and unsolved problems related to stability theory such as robust stability, bounded input - bounded output stability, internal stability,

root-locus, robust controllability, robust observability, etc.

Synchronization in chaotic dynamic systems has attracted increasing attention of scientists from various research fields for its advantages in practical application]. A wide variety of methods have been proposed for synchronization of chaotic systems, including linear feedback control [11], sliding mode control [12], adaptive control [13] and so on. Most of the methods mentioned above are used to guarantee the asymptotic stability of chaotic systems. Among the fractional order controllers, the fractional order active sliding mode control h (FOASMC) has been dealt more than others. In this paper, we introduce a fractional-order systems chaotic *chen*. To control and synchronization of chaotic fractional-order system a active sliding mode controller (ASMC) is proposed. This novel control law makes the system states asymptotically stable, simulation result show that the presented control method can easily eliminate chaos and stabilize the market. The rest of the paper is organized as follows.

## 2. FRACTIONAL-ORDER DERIVATIVE AND ITS APPROXIMATION

### A. Definition

The differ integral operator, represented by  ${}^0D_t^q$ , is a combined differentiation-integration operator commonly used in fractional calculus and general calculus operator, including fractional-order and integer is defined as:

$${}^0D_t^q = \begin{cases} \frac{d^q}{dt^q} & q > 0 \\ 1 & q = 0 \\ \int_0^t (d\tau)^{-q} & q < 0 \end{cases} \quad (1)$$

There are several definitions of fractional derivatives [14]. The best-known one is the Riemann-Liouville definition, which is given by

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau \quad (2)$$

Where n is an integer such that  $n - 1 < q < n$ ,  $\Gamma(0)$  is the Gamma function. The geometric and physical interpretation of the fractional derivatives was given as follows

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad (3)$$

The Laplace transform of the Riemann-Liouville fractional derivative is

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{q-1-k} f(t)}{dt^{q-1-k}}\right] \quad (4)$$

Where, L means Laplace transform, and s is a complex variable. Upon considering the initial conditions to zero, this formula reduces to

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} \quad (5)$$

The Caputo fractional derivative of order  $\alpha$  of a continuous function  $f : R^+ \rightarrow R$  is defined as follows

$$\frac{d^q f(t)}{dt^q} = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q-m+1}} d\tau & m-1 < q < m \\ \frac{d^m}{dt^m} f(t) & q = m \end{cases} \quad (6)$$

Thus, the fractional integral operator of order  $\alpha$  can be represented by the transfer function  $H(s) = \frac{1}{s^\alpha}$  in the frequency domain.

The standard definition of fractional-order calculus does not allow direct implementation of the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate fractional operators by using standard integer-order operators. In Ref.[15], an effective algorithm is developed to approximate fractional-order transfer functions, which has been adopted in [16] and has sufficient accuracy for time-domain implementations. In Table 1 of Ref [17], approximations for  $1/s^\alpha$  with  $\alpha$  from 0.1 to 0.9 in step 0.1 were given with errors of approximately 2 dB. We will use the  $1/s^{0.95}$  approximation formula [16] in the following simulation examples.

$$\frac{1}{s^{0.95}} \approx \frac{1.2831s^2 + 18.6004s + 2.0833}{1.2831s^3 + 18.4738s^2 + 2.6574s + 0.003} \quad (7)$$

In the simulation of this paper, we use approximation method to solve the fractional-order differential equations.

### 3. DESIGNING THE FRACTIONAL-ORDER ACTIVE SLIDING MODE CONTROL AND ANALYSIS

To design the active sliding mode controller, we have procedure a combination of the active controller and the sliding mode controller.

#### B. Active sliding mode controller design

Let us, consider a chaotic fractional-order description of the system as follows

$${}^0D_t^\alpha X_1 = A_1 X_1 + g_1(X_1) \quad \begin{matrix} 0 < \alpha \\ < 1 \end{matrix} \quad (8)$$

Where  $X_1(t) = (x_1, x_2, x_3)^T$  are real state vector,  $A_1 \in R^{3 \times 3}$  denotes the linear part of the system dynamics and  $g_1: R^3 \rightarrow R^3$  is nonlinear part of the system. Eq.(1) denotes the master system. Let  $X_0 = (x_{10}, x_{20}, x_{30})^T$  be the any initial conditions in the chaos attractor of fractional-orders system (8).

Now the controller  $u(t) \in R^3$  is added the slave system.

Thus:

$${}^0D_t^\alpha X_2 = A_2 X_2 + g_2(X_2) + u(t) \quad \begin{matrix} 0 < \alpha \\ < 1 \end{matrix} \quad (9)$$

That  $X_2, A_2$  and  $g_2$  imply the same roles as  $X_1, A_1$  and  $g_1$  for the master system. Synchronization of the systems means finding a control signal  $u(t) \in R^3$  that makes state of the slave system to evolves as the states of the master system.

Now we define errors dynamics as follows

$$\begin{aligned}
 0^{D_t^q} X_2 - 0^{D_t^q} X_1 &= A_2 X_2 + g_2(X_2) \\
 &- A_1 X_1 - g_1(X_1) \\
 &+ u(t)
 \end{aligned} \tag{10}$$

Thus:

$$\begin{aligned}
 0^{D_t^q} X_2 - 0^{D_t^q} X_1 &= A_2 X_2 + g_2(X_2) - A_1 X_1 \\
 &- g_1(X_1) \\
 + u(t)
 \end{aligned} \tag{11}$$

That:  $e = X_2 - X_1$  and  $A_1 = A_2 = A$

Now we assump:

$$\begin{aligned}
 G(X_1, X_2) &= g_2(x_2) - g_1(x_1) + \\
 &(A_2 \\
 - A_1)x_1
 \end{aligned} \tag{12}$$

The aim is to design the controller  $u(t) \in R^3$  such that:

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \| e(t) \| \\
 = 0
 \end{aligned} \tag{13}$$

Then use with the active control design procedure[19,20]

U(t) change as following:

$$\begin{aligned}
 &u(t) \\
 &= H(t) \\
 &- G(X_1, X_2)
 \end{aligned} \tag{14}$$

Eq.(14) describe the newly defined control input H(t).

Where  $H(t)$  is:

$$\begin{aligned}
 &H(t) \\
 &= Kw(t)
 \end{aligned} \tag{15}$$

Where  $k \in R^3$  is a constant gain vector and  $w(t) \in R$  is the control input that satisfies in:

$$\begin{aligned}
 W(t) \\
 = \begin{cases} w^+(t) & s(e) \geq 0 \\ w^-(t) & s(e) < 0 \end{cases}
 \end{aligned} \tag{16}$$

Where  $s = s(e)$  is a switching surface that describes the desired dynamics. The resultant error is then written by:

$$\begin{aligned}
 0^{D_t^q} e \\
 = Ae \\
 + KW(t)
 \end{aligned} \tag{17}$$

### C. Constructing a sliding surface

Constructing a sliding surface which represents a desired system dynamics and the sliding surface described as follows

$$\begin{aligned}
 s(e) \\
 = Ce
 \end{aligned} \tag{18}$$

Where  $C \in R^3$  is a constant vector. An equivalent control is found when  $\dot{S}(e) = 0$  which is a necessary condition for the state trajectory to stay on the switching surface  $S(e) = 0$ . Hence, the controlled system satisfies the following conditions in the steady state:

$$\begin{aligned}
 S(e) = 0 \quad \text{and} \quad \dot{S}(e) \\
 = 0
 \end{aligned} \tag{19}$$

Based on equation (17) to (19), It could be deduced:

$$\begin{aligned}
 \dot{S}(e) &= \left( C 0^{D_t^q} (Ae + kw(t)) \right) \\
 &= 0
 \end{aligned} \tag{20}$$

Thus,

$$\begin{aligned}
 & {}_0^{D_t^q} w(t) \\
 & = -(ck)^{-1} CA \left( {}_0^{D_t^q} e(t) \right) \quad (21)
 \end{aligned}$$

A solution of Eq.21. is:

$$\begin{aligned}
 & w_{eq}(t) \\
 & = -(CK)^{-1} CAe(t) \quad (22)
 \end{aligned}$$

#### D. Sliding mode control of fractional order system

We consider the constant plus proportional rate reaching law will be considered [18]. Accordingly the reaching law is obtained as:

$$\begin{aligned}
 & {}_0^{D_t^q} S \\
 & = -\rho \operatorname{sgn}(s) \\
 & - rs \quad (23)
 \end{aligned}$$

That  $\operatorname{sgn}(0)$  represents the sign function. The  $\rho, r$  are gains that the sliding conditions Eq.(19) is satisfied. From Eqs. (17), (18) have:

$$\begin{aligned}
 & {}_0^{D_t^q} S = C {}_0^{D_t^q} e \\
 & = C[Ae \\
 & + kw(t)] \quad (24)
 \end{aligned}$$

From Eqs. (23) and (24), find control effort can be defined as:

$$\begin{aligned}
 & w(t) = -(CK)^{-1} [C(rI + A)e \\
 & + \rho \operatorname{sgn}(s)]. \quad (25)
 \end{aligned}$$

#### E. Stability

First, we represent stability theorems from the fractional calculus.

**Theorem.1** (Matignon [19]). The following system:

$${}_0^{D_t^q} = Ax, \quad x(0) = x_0$$

Where  $0 < q < 1, x \in R^n$  and  $A \in R^{n \times n}$ , is asymptotically stable iff  $|\arg(\operatorname{eig} A)| > q\pi/2$ .

According to Theorem 1, as long as all eigenvalues of  $[A - K(CK)C(rI + A)]$  ( $\lambda_i = 1, 2, 3$ ) satisfy the conditions  $|\arg(\lambda_i)| > q\pi/2$ , the system is asymptotically stable.

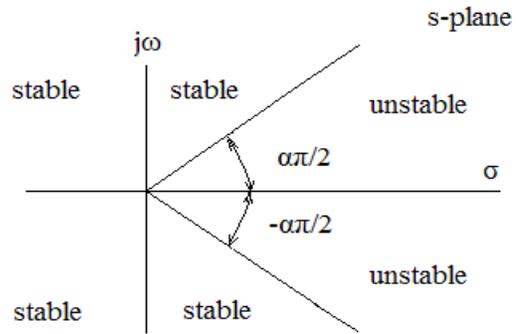


Fig.1. Stability region of linear fractional-order system with  $\alpha$

## 4. NUMERICAL SIMULATIONS

#### F. Synchronization between two fractional-order chen systems

The Lu system was introduced by Chen and Ueta in 1999 [20].

$$\begin{cases}
 {}_0^{D_t^q} x = \rho(y - x) \\
 {}_0^{D_t^q} y = -xz + vy \\
 {}_0^{D_t^q} z = xy - \mu z
 \end{cases} \quad (26)$$

For this system matrix A is

A

$$= \begin{bmatrix} -\rho & \rho & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$$

$$(27) \text{ Slave system } \begin{cases} 0^{D_t^{0.9}} x_1 = 35(y_2 - x_2) \\ 0^{D_t^{0.9}} y_2 = -x_2 z_2 + 28y_2 \\ 0^{D_t^{0.9}} z_2 = x_2 y_2 - 3z_2 \end{cases} \quad (29)$$

In this section, we consider using (ASMC) technique to obtain synchronization. This controller guarantees the synchronization two fractional orders Lu systems with The following initial conditions:

$$\begin{aligned} (x_{10}, y_{10}, z_{10}) &= (0, 3, 9) \\ &\text{and} \\ (x_{20}, y_{20}, z_{20}) &= (1, 4, -10). \end{aligned}$$

Consider two fractional order Lu systems as master and slave systems respectively:

$$\text{Master system } \begin{cases} 0^{D_t^{0.9}} x_1 = 35(y_1 - x_1) \\ 0^{D_t^{0.9}} y_1 = -x_1 z_1 + 28y_1 \\ 0^{D_t^{0.9}} z_1 = x_1 y_1 - 3z_1 \end{cases} \quad (28)$$

Parameters of the controller are chosen as  $k = [-1.9, -4.8, -4]^T$ ,  $C = [1, 1, -1]$ ,  $r = 66$  and  $\rho = 0.43$ . This selection of parameters results in eigenvalues  $[\lambda_1, \lambda_2, \lambda_3] = [-63, -65.4148, -6.5852]$  which located in a stable region ( $|\arg(\lambda_i)| > \alpha\pi/2$ ). Fig. shows the effectiveness of the proposed controller to synchronize two fractional-order modeled systems. It should be noted that control  $u(t)$ , has been activated at  $t = 0$ . The simulation results are shown in Fig.2.

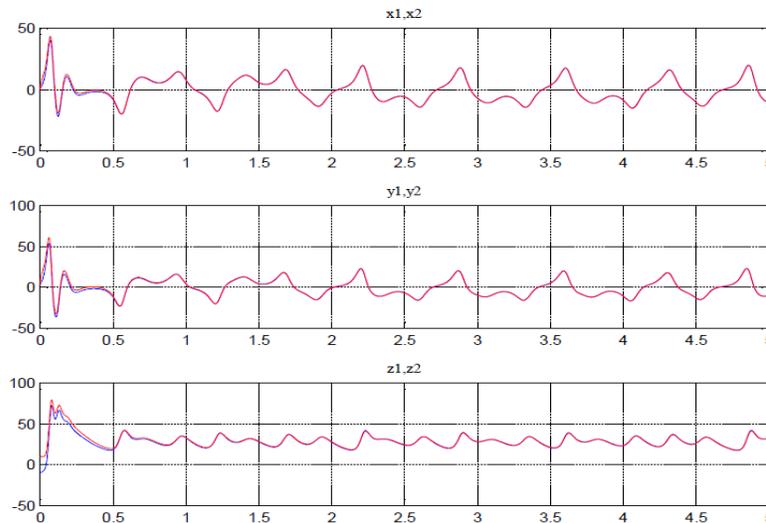
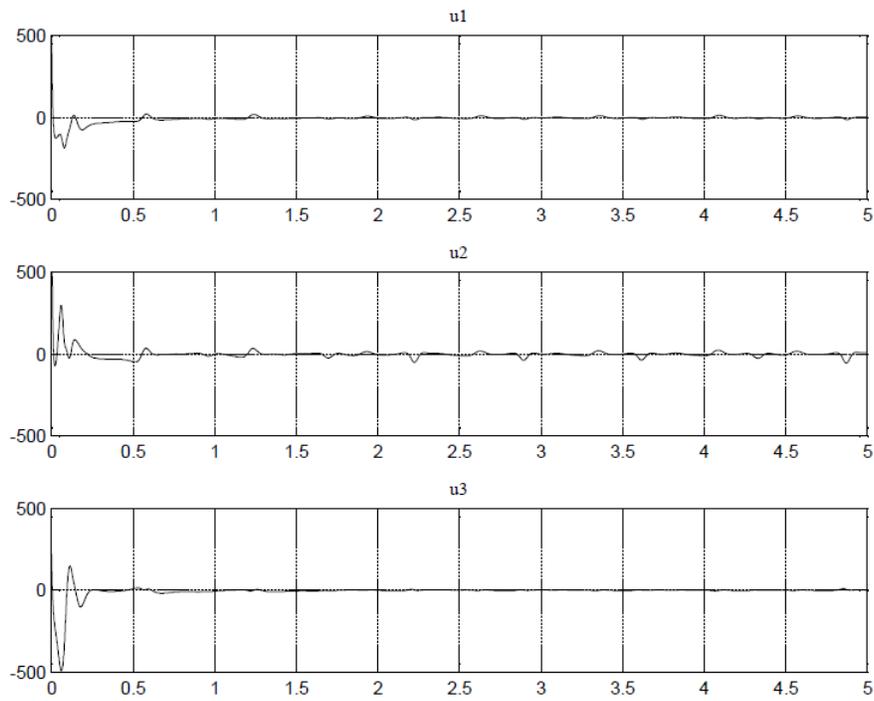
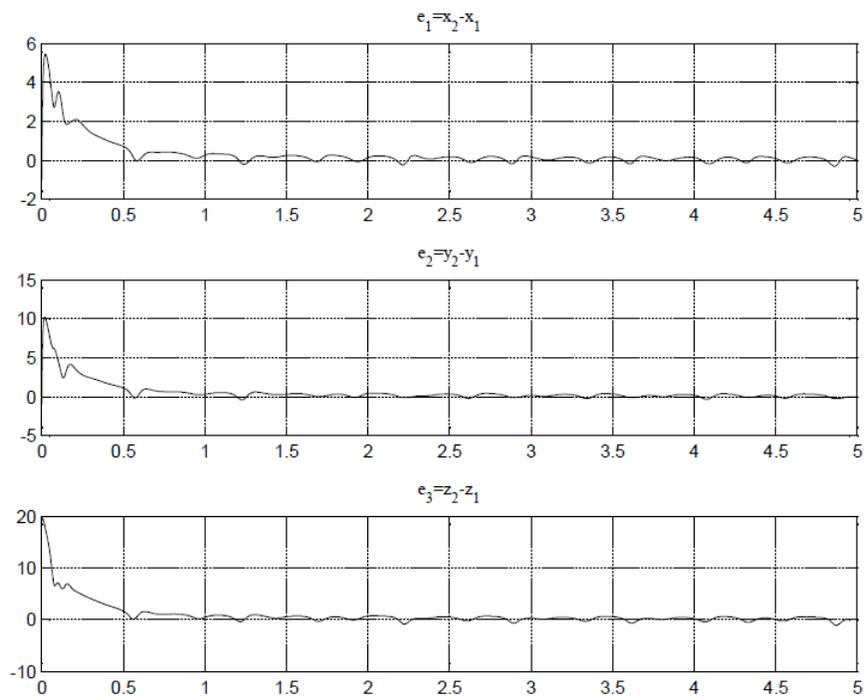


Fig.2.(a)



**Fig.2.(b)**



**Fig.2.(c)**

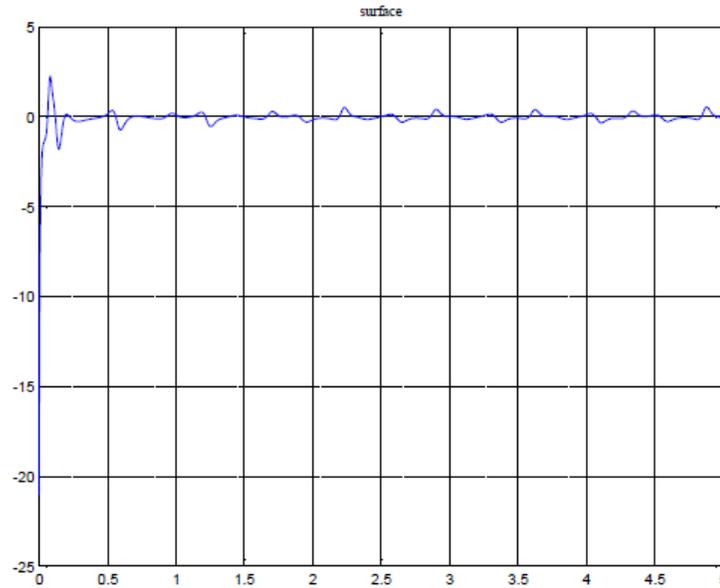


Fig.2.(d)

Fig.1. Results of simulating.

## 5. CONCLUSION

This paper we have studied numerical methods in fractional calculus. Then, we have represented the active sliding mode control to synchronize. The control parameters ( $r, k$  and  $c$ ), the master and slave systems are synchronized. Numerical simulations show the efficiency of the proposed controller to synchronize chaotic fractional-order.

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